H∞-Control: Overview

Basic Setup

Optimal H∞-control

- Find all stabilizing controllers \( \hat{K} \) such that \( \| T_{zw} \|_{\infty} \) is minimized

Suboptimal H∞-control

- For a given real number \( \gamma > 0 \), find all stabilizing controllers \( \hat{K} \) such that \( \| T_{zw} \|_{\infty} < \gamma \)

⇒ We will study the suboptimal problem
$H_\infty$-Control: General Case

**Plant**

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

**Transfer Functions**

$$G_{11} = C_1(sI - A)^{-1}B_1 + D_{11}$$
$$G_{12} = C_1(sI - A)^{-1}B_2 + D_{12}$$
$$G_{21} = C_2(sI - A)^{-1}B_1 + D_{21}$$
$$G_{22} = C_2(sI - A)^{-1}B_2 + D_{22}$$

**Remark**

- Only $G_{22}$ in feedback loop with controller $\hat{K}$
Duality and Special Problems: Duality

**Dual Feedback Loops**

Implications

- Transfer matrices: $T_{zw}^T = T_{\tilde{z}\tilde{w}}$
- Internal stability: $\hat{K}$ stabilizes the feedback loop with $G$ if and only if $\hat{K}^T$ stabilizes the feedback loop with $G^T$

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Duality and Special Problems: Full Information

**Plant**

$$G_{FI}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

**Properties**

- Controller gets the plant output signal
  
  $$ y = C_2 x + D_{12} \ w + D_{22} \ u = \begin{bmatrix} x \\ w \end{bmatrix} $$

  $\Rightarrow$ Controller has full information about the plant state $x$

  $\Rightarrow$ Controller has full information about the disturbance input $w$
Duality and Special Problems: Full Control

Plant

\[
G_{FC}(s) = \begin{bmatrix}
A & B_1 & I & 0 \\
C_1 & D_{11} & 0 & I \\
C_2 & D_{21} & 0 & 0
\end{bmatrix}
\]

Properties

- Controller has full access to the plant state
  \[ \dot{x} = Ax + B_1 w + u \]
- Controller has full access to the plant output \( z \)
  \[ z = C_1 x + D_{11} w + u \]

⇒ Full information and full control are dual problems: \( G_{FI}^T = G_{FC} \)

Duality and Special Problems: Disturbance Feedforward

Plant

\[
G_{DF}(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & I & 0
\end{bmatrix}
\]

Properties

- Plant output \( y \) directly depends on the disturbance input \( w \)
  \[ y = C_2 x + w \]

Equivalence to FI

- If controller \( \hat{K}_{DF} \) internally stabilizes disturbance feedforward problem, then controller \( \hat{K}_{FI} = \hat{K}_{DF} [C_2 I] \) stabilizes the full information problem
- More information in Chapter 12 of "Robust and Optimal Control"
Duality and Special Problems: Disturbance Feedforward

Computation

\[ G_{OE}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{bmatrix} \]

Properties
- Plant output \( z \) directly depends on control input \( u \)
  \[ z = C_1 x + D_{11} w + u \]

Equivalence and Duality
- Output estimation problem is dual to disturbance feedforward problem
  \[ G_{OE}^T = G_{DF} \]
- Output estimation problem is equivalent to full control problem
Duality and Special Problems: Basic Assumption

**Simplification** $D_{22} = 0$

⇒ We can assume without loss of generality that $D_{22} = 0$
⇒ If $D_{22} \neq 0$, we can use the controller $\hat{K} (I + D_{22} \hat{K})^{-1}$

**Inner Functions: Overview**

**Definition**
- Notation for transfer matrix $P$: $P\sim(s) = P^T(-s)$
- A transfer matrix $P \in \mathcal{RH}_\infty$ is called inner if $P\sim P = I$

**Lemma (Inner Transfer Matrix)**

Consider the system in lower LFT with the transfer matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{RH}_\infty$ and the proper rational transfer matrix $Q$. Assume that $P$ is inner and $P^{-1}_{21} \in \mathcal{RH}_\infty$. The system is internally stable, well-posed and $\|T_{zw}\|_\infty < 1$ if and only if $Q \in \mathcal{RH}_\infty$ and $\|Q\|_\infty < 1$. 
Inner Functions: Proof

Computation

Algebraic Riccati Equations: Definition

Components
- Real matrices: $A, Q, R \in \mathbb{R}^{n \times n}$
- $Q, R$ symmetric

Algebraic Riccati Equation (ARE)

$$A^*X + XA + XR + Q = 0$$

⇒ Solution matrix $X \in \mathbb{R}^{n \times n}$

Formulation using Hamilton Matrix $H$

$$\begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = 0$$

⇒ Study Hamilton matrix $H$ instead of original ARE
Algebraic Riccati Equations: Invariant Subspaces

**Definition (Invariant Subspace)**

Let $T : \mathcal{W} \rightarrow \mathcal{W}$ be a linear mapping from a vector space $\mathcal{W}$ to itself. A subspace $\mathcal{V} \subseteq \mathcal{W}$ is a $T$-invariant subspace of $\mathcal{W}$ if for every $v \in \mathcal{V}$, $Tv \in \mathcal{V}$.

**Properties**

- For a matrix (linear mapping) $T \in \mathbb{C}^{2n \times 2n}$, an $T$-invariant subspace of $\mathbb{C}^{2n}$ of dimension $n$ is always spanned by $n$ (generalized) eigenvectors of $T$.

**Example**

Gap 8

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Algebraic Riccati Equations: Example

**Computation**

Gap 9

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Lemma

Let $H \in \mathbb{C}^{2n \times 2n}$ be a Hamilton matrix and assume that $V \subseteq \mathbb{C}^{2n}$ is an $n$-dimensional $H$-invariant subspace such that $V = \text{image} \left( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right)$ with $X_1, X_2 \in \mathbb{C}^{n \times n}$. If $X_1$ is invertible, then $X := X_2 X_1^{-1}$ is a solution of the ARE.

Algebraic Riccati Equations: Stable Solutions

Special Case

- $H$ has no eigenvalues on the imaginary axis
- $n$ eigenvalues in OLHP and $n$ eigenvalues in ORHP
- Invariant subspace $V_-$ that is spanned by (generalized) eigenvectors that correspond to eigenvalues of $H$ in the OLHP
- $V_- = \text{image} \left( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right)$
- Unique solution of the ARE: $X := X_2 X_1^{-1}$

Notation

- Set of all Hamilton matrices without eigenvalues on the imaginary axis: $\text{dom}(\text{Ric})$
- **Stable solution** for $H \in \text{dom}(\text{Ric})$: $X = \text{Ric}(H)$
  - Eigenvalues of $A + RX$ are the OLHP eigenvalues of $H$
Algebraic Riccati Equations: Example

Gap 11

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Algebraic Riccati Equations

Algebraic Riccati Equations: Example

Gap 12

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